NAME:

Solutions To Math 155 Practice Exam 1

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. Find $(f^{-1})'(2)$ given that $f(x) = x^3 + 3\sin x + 2\cos x$. [10 pts]

Solution: $f(0) = 2$, therefore $(f^{-1})(2) = 0$. Observe that the derivative $(f^{-1})'(2)$ represents the slope of the tangent line to the curve $y = f(x)$ at the point (0, 2) when y is the horizontal axis and x is the vertical axis. Thus, $(f^{-1})'(2) = \frac{1}{f'(0)} = \frac{1}{3}$.

2. Differentiate $y = \frac{(x^3+2x^2)^4 \sec^2 x}{x^{1/3}}$ [Hint: Be smart about it!] [10 pts]

Solution: $\ln y = 4[2\ \ln x + \ln(x+2)] + 2\ln(\sec x) - \frac{1}{3}\ln x$. Thus, $\frac{1}{y}\frac{dy}{dx} = \frac{8}{x} + \frac{4}{x+2} + 2\frac{\sec x \tan x}{\sec x} - \frac{1}{3x} = \frac{8}{x} + \frac{4}{x+2} + 2\tan x - \frac{1}{3x}$. In particular, $\frac{dy}{dx} = \frac{(x^3+2x^2)^4 \sec^2 x}{x^{1/3}}\left(\frac{8}{x} + \frac{4}{x+2} + 2\tan x - \frac{1}{3x}\right)$.

3. Differentiate $y = (\tan x)^{sin\, x}$ [10 pts]

Solution: $\ln y = \sin x\ \ln(\tan x)$. $\frac{1}{y}\frac{dy}{dx} = \cos x\ \ln(\tan x) + \sin x\ \frac{\sec^2 x}{\tan x}$. In particular, $\frac{dy}{dx} = (\tan x)^{sin\, x}\left(\cos x\ \ln(\tan x) + \sin x\ \frac{\sec^2 x}{\tan x}\right)$.

4. Evaluate the integral $\int \frac{2^x}{2^x+1}dx$ [10 pts]

Solution: $\int \frac{2^x}{2^x+1}dx = \frac{1}{\ln 2}\int \frac{2^x \ln 2}{2^x+1}dx = \frac{1}{\ln 2}\ln(2^x+1) + C$

5. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
(a) Find the mass that remains after t years. [6 pts]

Solution: $M(t) = 100\left(\frac{1}{2}\right)^{t/30} = 100e^{\ln\left[\left(\frac{1}{2}\right)^{t/30}\right]} = 100e^{\frac{-\ln 2}{30}t}$.

(b) How much of the sample remains after 100 years? [2 pts]

Solution: $M(100) = 100e^{\frac{-\ln 2}{30}100} \approx 9.92$ mg.

(c) After how long will only 1 mg remain? [2 pts]

Solution: $1 = M(T) = 100e^{\frac{-\ln 2}{30}T}$. Solving for T, we obtain $\ln\left(\frac{1}{100}\right) = \frac{-\ln 2}{30}T$ or $T = 30\frac{\ln 100}{\ln 2} \approx 199.3$ years.

6. Evaluate $\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2}$ [10 pts]

Solution: $\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{4}\int_0^{\sqrt{3}/4} \frac{4dx}{1+(4x)^2} = \frac{1}{4}\tan^{-1}(4x)\Big|_0^{\sqrt{3}/4} = \frac{1}{4}\tan^{-1}(\sqrt{3}) = \frac{\pi}{12}$

7. Compute $\lim_{x\to\infty}(e^x + x)^{1/x}$ [10 pts]

Solution:

$$\lim_{x\to\infty}(e^x + x)^{1/x} = \lim_{x\to\infty} e\left(1 + \frac{x}{e^x}\right)^{1/x} = \lim_{x\to\infty} e\left[\left(1 + \frac{x}{e^x}\right)^{e^x/x}\right]^{1/e^x} = e$$

Alternatively, use l'Hospital's Rule:

$$\lim_{x\to\infty}(e^x + x)^{1/x} = \lim_{x\to\infty} e^{\frac{\ln[e^x+x]}{x}} = \lim_{x\to\infty} e^{\left(\frac{e^x+1}{e^x+x}\right)} = \lim_{x\to\infty} e^{\left(\frac{e^x}{e^x+1}\right)} = \lim_{x\to\infty} e^{1-\frac{1}{e^x+1}} = e$$

8. Evaluate the integral $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\, dr$. [10 pts]

Solution: Integrating by parts, we obtain that $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\, dr = \int_0^1 \frac{r}{\sqrt{4+r^2}} r^2 dr = r^2\sqrt{4+r^2}\Big|_0^1 - \int_0^1 2r\sqrt{4+r^2}dr = \sqrt{5} - \frac{2}{3}(4+r^2)^{3/2}\Big|_0^1 = \sqrt{5} - \frac{10}{3}\sqrt{5} + \frac{16}{3} = \frac{16}{3} - \frac{7}{3}\sqrt{5}$

9. Evaluate the integral $\int_1^4 e^{\sqrt{x}}dx$. [10 pts]

Solution: Let $u = \sqrt{x}$. Then $u^2 = x$ and $2u\, du = dx$. In particular, $\int_1^4 e^{\sqrt{x}}dx = \int_1^2 2u\, e^u du$. Integrating by parts, we obtain $\int_1^2 2u\, e^u du = 2u\, e^u\big|_1^2 - \int_1^2 2e^u du = 4e^2 - 2e - 2e^2 + 2e = 2e^2$.

10. Find $\int \tan^6 x \sec^4 x\, dx$. [10 pts]

Solution: Since the derivative of $\tan x$ is $\sec^2 x$, we shall attempt to reduce the power of $\sec^4 x$ to $\sec^2 x$:

$$\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x \sec^2 x\ \sec^2 x\, dx = \int \tan^6 x\,(1+\tan^2 x)\sec^2 x\, dx$$
$$= \int \tan^6 x \sec^2 x\, dx + \int \tan^8 x \sec^2 x\, dx = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C$$

Extra-Credit

11. Evaluate $\lim_{x\to 0}(\cos x - \sin 2x)^{1/x}$ [10 pts]

Solution: $\lim_{x\to 0}(\cos x - \sin 2x)^{1/x} = \lim_{x\to 0} e^{\left(\frac{\ln[\cos x - \sin 2x]}{x}\right)} = e^{\frac{-\sin x - 2\cos 2x}{\cos x - \sin 2x}} = e^{-2}$. This may be observed directly without using l'Hospital... Open your mind!

12. Find the infinite polynomial expansion for $f(x) = \tan^{-1} x$.
[10 pts]

Solution: Observe that $f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$ (why?). Thus $f(x) = \int \sum_{n=0}^{\infty}(-x^2)^n\,dx = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

13. Suppose that some function f has the following properties. $f(x+y) = f(x) + f(y) + 5x^2y + 5y^2x$ and $\lim_{x\to 0}\frac{f(x)}{x} = -3$. Find $f'(x)$.

[10 pts]

Solution: $\frac{f(x+y)-f(x)}{y} = \frac{f(y)+5x^2y+5y^2x}{y} = \frac{f(y)}{y} + 5x^2 + 5yx$. Thus $f'(x) = \lim_{y\to 0}\frac{f(x+y)-f(x)}{y} = \lim_{y\to 0}\left(\frac{f(y)}{y} + 5x^2 + 5yx\right) = -3 + 5x^2$.